Time dependence of moments of an exactly solvable Verhulst model under random perturbations *

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Abstract

Explicit expressions for one point moments corresponding to stochastic Verhulst model driven by Markovian coloured dichotomous noise are presented. It is shown that the moments are the given functions of a decreasing exponent. The asymptotic behavior (for large time) of the moments is described by a single decreasing exponent.

Keywords: Stochastic Verhulst model, one point moments, explicit expressions.

1 Introduction

There are a lot of papers devoted to description of the temporary evolution of moments with exactly solvable nonlinear stochastic equations. In [1] we gave some general procedure to explicitly solve the master equations of hyperbolic type corresponding to nonlinear stochastic equations driven by

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dichotomous noise. The method is based on a generalization of Laplace factorization method [2, 3]. As an example we have considered complete exact nonstationary solution of the master equations for probability distribution corresponding to stochastic Verhulst model.

In this paper we calculate one points moments of arbitrary degree and discuss its time evolution. Let us consider nonlinear stochastic dynamical system

$$\dot{x} = p(x) + \alpha(t)q(x), \tag{1}$$

where $x(t)$ is the dynamical variable, $p(x)$, $q(x)$ are given functions of x, $\alpha(t)$ is the random function with known statistical characteristics. The model (1) arises in different applications (see for example [4, 5] and bibliography therein). An important application of this model consists in study of noise-induced transitions in physics, chemistry and biology. The functions $p(x)$, $q(x)$ are often taken polynomial. For example, if we set $p(x) = p_1x + p_2x^2$, $q(x) = q_2x^2$, $p_1 > 0$, $p_2 < 0$, $|p_2| > q_2 > 0$, then the equation (1) describes the population dynamics when resources (nutrition) fluctuate (Verhulst model). In the following we will assume $\alpha(t)$ to be binary (dichotomic) noise $\alpha(t) = \pm 1$ with switching frequency $2\nu > 0$. As one can show (see [6]), the averages $W(x,t) = \langle \widetilde{W}(x,t)\rangle$ and $W_1(x,t) = \langle \alpha(t)\widetilde{W}(x,t)\rangle$ for the probability density $\widetilde{W}(x,t)$ in the space of possible trajectories $x(t)$ of the dynamical system (1) satisfy a system (also called "master equations"):

$$\begin{cases} W_t + (p(x)W)_x + (q(x)W_1)_x = 0, \\ (W_1)_t + 2\nu W_1 + (p(x)W_1)_x + (q(x)W)_x = 0. \end{cases} \tag{2}$$

We suppose that the initial condition $W(x,0) = W_0(x)$ for the probability distribution is nonrandom. This implies that the initial condition for $W_1(x,t)$ at $t = 0$ is zero: $W_1(x,0) = \langle \alpha(0)\widetilde{W}(x,0)\rangle = \langle \alpha(0)\rangle W_0(x) = 0$. The probability distribution $W(x,t)$ should be nonnegative and normalized for all t: $W(x,t) \geq 0$, $\int_{-\infty}^{\infty} W(x,t)\, dx \equiv 1$.

In [1] we have obtained the following explicit form of the complete solution of the system (2) for probability distribution $W(x,t)$:

$$W(x,\tau) = \tfrac{1}{2}e^{-\tau}\left\{\delta\left(x - \tfrac{e^{-\tau}x_*}{1+(p_2+q_2)(e^{\tau}-1)x_*}\right) + \delta\left(x - \tfrac{e^{-\tau}x_*}{1-(p_2-q_2)(e^{\tau}-1)x_*}\right)\right\} +$$
$$\tfrac{1}{2q_2x^2}\left\{H\left(\tfrac{x}{e^{\tau}(1+(p_2-q_2)-x(p_2-q_2))} - x_*\right) - H\left(\tfrac{x}{e^{\tau}(1+(p_2+q_2)-x(p_2+q_2))} - x_*\right)\right\}, \tag{3}$$

where $\tau = \nu t$ is the dimensionless time and x_* is an initial value for (1), $H(z) = \int_{-\infty}^{z} \delta(\theta)d\theta$ is the Heaviside function.

The solution (3) corresponds to Cauchy problem $W_0(x) = \delta(x - x_*)$. Here we set that $\nu = 1$. The function $W(x,\tau)$ is in fact a conditional probability

distribution, that is $W(x,\tau)\Delta x \equiv W(x,\tau \mid x_*, \tau=0)\Delta x$ is the probability that at the time τ the dynamical variable x belongs to interval $(x, x+\Delta x)$ under condition that at some previous initial time $\tau = 0$ the variable x is equal to x_*.

From the equation (1) follows that the dynamical variable has three stationary points:

$$x_1 = \frac{1}{|p_2| + q_2}, \quad x_2 = \frac{1}{|p_2| - q_2}, \quad x_3 = 0.$$

It is convenient to use the definition x_1 and x_2 for transformation of the expression (3) to the form:

$$W(x,\tau) = \frac{1}{2}e^{-\tau}\left\{\delta\left(x - \frac{e^\tau x_* x_2}{x_2 + (e^\tau - 1)x_*}\right) + \delta\left(x - \frac{e^\tau x_* x_1}{x_1 + (e^\tau - 1)x_*}\right)\right\} +$$
$$\frac{x_1 x_2}{(x_2 - x_1)x^2}\left\{H\left(\frac{x x_1 e^{-\tau}}{x_1 + x(e^{-\tau} - 1)} - x_*\right) - H\left(\frac{x x_2 e^{-\tau}}{x_2 + x(e^{-\tau} - 1)} - x_*\right)\right\}. \tag{4}$$

2 Calculation of one point moments

The one point conditional moments of n-th order one defines as

$$\kappa_n(\tau) = \langle x^n(\tau) | x(0) = x_*, \tau = 0\rangle = \int_{(D)} x^n W(x,\tau) dx, \tag{5}$$

where (D) is the support of the probability distribution. Further we consider the case $(D) = (x_1, x_2)$. After simple calculations one obtains

$$\kappa_n(\tau) = \frac{1}{2}e^{-\tau}\left\{\left(\frac{e^\tau x_* x_2}{x_2 + (e^\tau - 1)x_*}\right)^n + \left(\frac{e^\tau x_* x_1}{x_1 + (e^\tau - 1)x_*}\right)^n\right\} +$$
$$\frac{x_1 x_2}{(x_2 - x_1)(n-1)}\left\{(x_2\beta(\tau))^{n-1} - (x_1\gamma(\tau))^{n-1}\right\}, \tag{6}$$

where

$$\beta(\tau) = \frac{x_*}{x_* + (x_2 - x_*)e^{-\tau}},$$
$$\gamma(\tau) = \frac{x_*}{x_* - (x_* - x_1)e^{-\tau}}.$$

Let $\tau \to 0$, then $\beta(\tau) \to \frac{x_*}{x_2}$ and $\gamma(\tau) \to \frac{x_*}{x_1}$. In this limit from (6) one has $\kappa_n(\tau) \to x_*^n$. Let us consider another asymptotic $\tau \to \infty$. From (6) one obtains for $n = 1$ the following stationary value of the moment

$$\kappa_1(\tau) = \frac{x_1 x_2}{x_2 - x_1}(\ln x_2 - \ln x_1),$$

and for $n \neq 1$ stationary values of moments are equal to

$$\kappa_n(\tau) = \frac{x_1 x_2}{n-1}\left(x_2^{n-2} + x_2^{n-3}x_1 + ... + x_2 x_1^{n-3} + x_1^{n-2}\right).$$

Generally the moments $\kappa_n(\tau)$ are given functions depending on the decreasing exponent $e^{-\tau}$ and can be represented by a series over the powers of $e^{-\tau}$. In [8] a similar representation was found for the stochastic Verhulst model with fluctuating coefficient at the first degree of the dynamical variable x. In the limit for large $\tau \gg 1$ the time behavior of $\kappa_n(\tau)$ can be described by a single exponent.

Important role is played by the first two initial moments ($n = 1, 2$). Let us consider the moment of first order. In this case

$$\kappa_1(\tau) = \frac{x_*}{2}\left(\frac{1}{1 + (e^\tau - 1)\frac{x_*}{x_2}} + \frac{1}{1 + (e^\tau - 1)\frac{x_*}{x_1}}\right) + \frac{x_1 x_2}{x_2 - x_1} \ln \frac{x_2 \beta(\tau)}{x_1 \gamma(\tau)}. \quad (7)$$

In the asymptotics $\tau \to \infty$ from (7) in first order over the infinitesimal $\exp(-\tau)$, one has

$$\kappa_1(\tau) \approx \frac{x_1 x_2}{x_2 - x_1} \ln \frac{x_2}{x_1} + \left(\frac{x_1 + x_2}{2} - \frac{x_1 x_2}{x_*}\right) e^{-\tau}. \quad (8)$$

It is interesting that there exists the initial value $x_* = \frac{2x_1 x_2}{x_1 + x_2} \equiv \frac{1}{|p_2|}$. In this case the coefficient at $\exp(-\tau)$ is equal to zero. Therefore we should take into account the next order, i.e. $\exp(-2\tau)$. Physically it means that in point $x_* = \frac{1}{|p_2|}$ the correlation of variable $x(\tau)$ with the given initial value of variable x ($x(0) = x_*$) decreases more rapidly at $\tau \to \infty$. When $x_* \neq \frac{1}{|p_2|}$ the correlations tends to stationary level as $\exp(-\tau)$.

Here we give also an explicit expression for the case $n = 2$:

$$\kappa_2(\tau) = \frac{1}{2}e^{-\tau}\left[\left(\frac{e^\tau x_* x_2}{x_2 + (e^\tau - 1)x_*}\right)^2 + \left(\frac{e^\tau x_* x_1}{x_1 + (e^\tau - 1)x_*}\right)^2\right] + \frac{x_*^2 x_1 x_2 (1 - e^{-\tau})}{(x_* + (x_2 - x_*)e^{-\tau})(x_* - (x_* - x_1)e^{-\tau})}. \quad (9)$$

3 Concluding remarks

We have considered the time evolution of one point moments of dynamical variable corresponding to the stochastic Verhulst model. The explicit form of the moments shows that the moments are the functions of $\exp(-\tau)$. In [8] it was shown for Verhulst model when parameter fluctuates at dynamical variable x (not x^2), that the exact solution for one point moments is presented by a series over powers of $\exp(-\tau)$. From formulae obtained here one can write the moments $\kappa_n(\tau)$ in the same form. It should be noted that in this communication we have obtained an explicit form of the solution. Under the condition $\tau \to \infty$ the moments decrease in time as single exponential function. It is shown that the time dependence of the moment $\kappa_1(\tau)$ which physically describes the correlation between the value of the dynamical variable x at the time τ with its given (nonrandom value x_*) at the initial time $\tau = 0$ changes. This time behavior depends on the choice of the initial value x_*. The critical value is $x_* = 1/|p_2|$. For this value of x_* in the limit $\tau \to \infty$ the correlations decrease as $\exp(-2\tau)$, not as $\exp(-\tau)$.

It should be noticed that the one-point moments for some special type of the dynamical system (1) with polynomial functions $p(x)$ and $q(x)$ Gaussian white noise fluctuation coefficient at the first power of x were considered in [7, 9, 10], where it was shown that the asymptotic behavior of the moments is described by a power function.

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